8. Gradients

8.1 Acyclic Partial Matchings

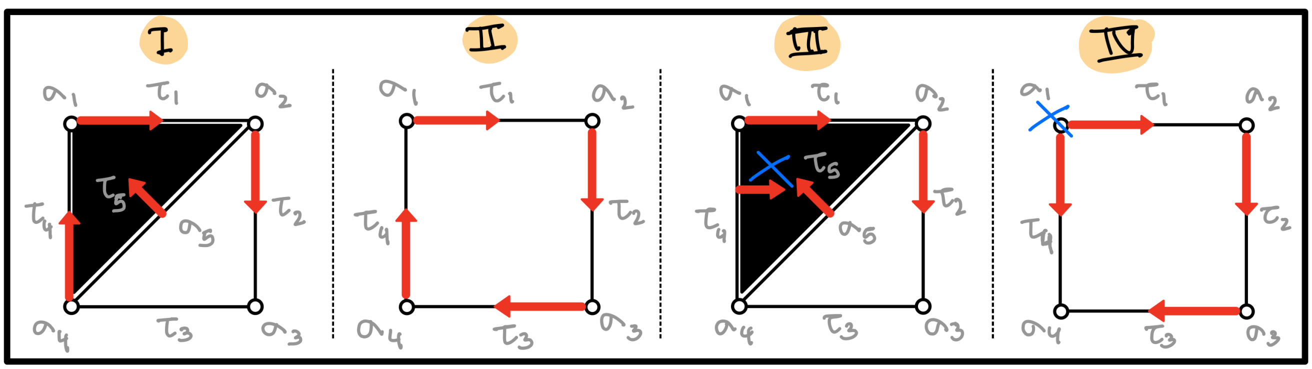
Let K be a simplicial complex. For any pair of simplices σ, τ in K, we write $\sigma \lhd \tau$ to indicate that σ is a codimension one face of τ, i.e., that $\sigma \leq \tau$ and $\dim \tau - \dim \sigma = 1$.

Definition 8.1. A **partial matching** on K is a collection $\Sigma = \{(\sigma_\bullet \lhd \tau_\bullet)\}$ of simplex-pairs in K subject to the following constraint: if a pair $(\sigma \lhd \tau)$ lies in Σ, then neither σ nor τ appear in any other pair of Σ.

More elaborately, a partial matching Σ consists of two disjoint subsets of simplices $S_\Sigma, T_\Sigma \subset K$ along with a bijection $\mu_\Sigma : S_\Sigma \xrightarrow{\sim} T_\Sigma$ so that $\sigma \lhd \mu(\sigma)$ holds for every σ in S_Σ. Crucially, we do not require $K = S_\Sigma \cup T_\Sigma$, so there might be simplices in K which remain untouched by the matching. These unmatched simplices lying in the complement $C_\Sigma := K - (S_\Sigma \cup T_\Sigma)$ are called **Σ-critical**. It should also be noted that none of the sets S_Σ, T_Σ and C_Σ are required by this definition to be subcomplexes of K.

Partial matchings are relevant to us because under certain assumptions (to be described in gory detail below), we can compute the homology groups of K using a chain complex whose chain groups are built using *only the critical simplices* of a partial matching. Thus, finding a good partial matching with very few critical simplices makes it possible to drastically reduce the algorithmic burden of computing homology groups. Before describing all this machinery, we will examine some examples (and non-examples) of partial matchings.

Example 8.2. Partial matchings are usually illustrated using arrows pointing from the smaller simplex σ to the larger simplex τ whenever $(\sigma \lhd \tau)$ lies in Σ. Consider the diagrams **I-IV** below:



Both **I** and **II** constitute legal partial matchings — the elements of S_Σ are sources of arrows while the elements of T_Σ are targets. The simplices σ_3 and τ_3 in **I** remain untouched by arrows and are therefore critical (but note that **II** has no critical simplices). Neither **III** nor **IV** are partial matchings — in **III** there is a simplex with two incoming arrows whereas in **IV** there is a simplex with two outgoing ones.

Fix a partial matching Σ on K.

Definition 8.3. A **Σ-path** is a zigzag sequence of distinct simplices in K of the form

$$\rho = (\sigma_1 \lhd \tau_1 \rhd \sigma_2 \lhd \tau_2 \rhd \cdots \rhd \sigma_m \lhd \tau_m), \tag{7}$$

where $(\sigma_i \lhd \tau_i)$ lies in Σ for all i in $\{1, \ldots, m\}$. Such a path is **gradient** if either $m = 1$ or σ_1 is not a face of τ_m. We say that Σ is an **acyclic** partial matching if all of its paths are gradient.

Of the two legal partial matchings depicted in Example 8.2 above, only **I** is acyclic — the non-gradient paths in **II** can be discovered by starting at any vertex and following arrows until the loop is completed. Henceforth we will only consider acyclic partial matchings; our interest in this special subset is primarily motivated by the following result.

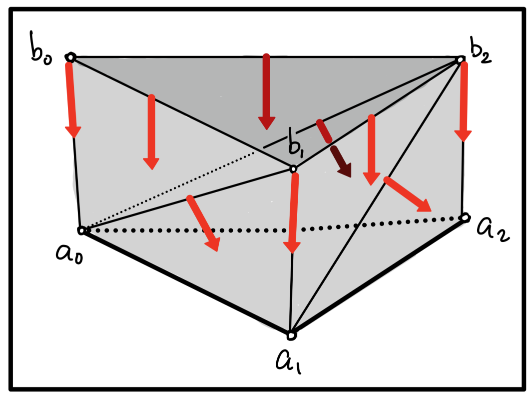
THEOREM 8.4. *Let Σ be an acyclic partial matching on a simplicial complex K, and let $\mathbb{F}$ be any coefficient field. There exists a chain complex (of $\mathbb{F}$-vector spaces)*

$$\cdots \xrightarrow{d^\Sigma_{k+1}} \mathbf{C}^\Sigma_k(K;\mathbb{F}) \xrightarrow{d^\Sigma_k} \mathbf{C}^\Sigma_{k-1}(K;\mathbb{F}) \xrightarrow{d^\Sigma_{k-1}} \cdots \xrightarrow{d^\Sigma_2} \mathbf{C}^\Sigma_1(K;\mathbb{F}) \xrightarrow{d^\Sigma_1} \mathbf{C}^\Sigma_0(K;\mathbb{F}) \longrightarrow 0$$

satisfying three properties:

(1) *each chain group $\mathbf{C}^\Sigma_k(K;\mathbb{F})$ is $\bigoplus_\alpha \mathbb{F}$, indexed by critical k-simplices $\alpha \in C_\Sigma$,*
(2) *the boundary operators d^Σ_k are explicitly determined by knowledge of Σ-paths, and*
(3) *the homology groups of $(\mathbf{C}^\Sigma_\bullet(K;\mathbb{F}), d^\Sigma_\bullet)$ are isomorphic to those of K.*

The next two Sections are devoted to the task of building the boundary operators $d^\Sigma_\bullet$ from Σ-paths and proving the isomorphism on homology as promised by properties (2) and (3) respectively. If the set of critical simplices $C_\Sigma \subset K$ forms a subcomplex of K, then the Theorem above can be proved without much difficulty. The illustration here contains one example of this easy case: the complex K is a triangulation of the cylinder $\partial\Delta(2) \times [0,1]$, and the critical simplices C_Σ consist of the base circle (spanned by the vertices a_0, a_1, a_2 and the three edges between them). In this case there is a sequence of elementary collapses (as in Proposition 2.14) from K to C_Σ. This establishes a homotopy equivalence, and hence the desired isomorphisms on homology by Theorem 4.24. Thus, our challenge in proving Theorem 8.4 stems from the fact that in general $C_\Sigma \subset K$ will not be a subcomplex.



REMARK 8.5. Acyclic partial matchings are combinatorial analogues of *gradient vector fields* from differential geometry, and the main idea behind the proof of Theorem 8.4 is to deform the original chain complex $(\mathbf{C}_\bullet(K), \partial^K_\bullet)$ to the smaller chain complex $(\mathbf{C}^\Sigma_\bullet(K), d^\Sigma_\bullet)$ by flowing down along the arrows of this combinatorial gradient vector field. As such, Theorem 8.4 forms the simplicial analogue of one of the main results from smooth *Morse theory*. For these historical reasons, $(\mathbf{C}^\Sigma_\bullet(K), d^\Sigma_\bullet)$ is called the **Morse chain complex** associated to Σ, and the study of acyclic partial matchings is called **discrete Morse theory**.

8.2 THE MORSE CHAIN COMPLEX

Let K be a simplicial complex with ordered vertices. Given any simplices σ and τ in K, let $[\tau : \sigma] \in \{0, \pm 1\}$ indicate the coefficient of σ in the boundary of τ (see Definition 3.4) — this number is nonzero if and only if $\sigma \lhd \tau$. Fix an acyclic partial matching Σ on K as in Definition 8.3. Here we will build the boundary operators $d^\Sigma_\bullet$ whose existence was promised in the statement of Theorem 8.4. The first step in this direction is to associate an algebraic contribution to each Σ-path.

DEFINITION 8.6. The **weight** $w(\rho) \in \{\pm 1\}$ of the Σ-path

$$\rho = (\sigma_1 \lhd \tau_1 \rhd \sigma_2 \lhd \tau_2 \rhd \cdots \rhd \sigma_m \lhd \tau_m),$$

is defined to be the product

$$w(\rho) = \frac{-1}{[\tau_1 : \sigma_1]} \cdot [\tau_2 : \sigma_1] \cdot \frac{-1}{[\tau_2 : \sigma_2]} \cdots [\tau_{m-1} : \sigma_m] \cdot \frac{-1}{[\tau_m : \sigma_m]}$$

One can equivalently collect numerators and denominators to express the weight of each Σ-path ρ as a single ratio

$$w(\rho) = (-1)^m \cdot \frac{\prod_{i=1}^{m-1}[\tau_i : \sigma_{i+1}]}{\prod_{i=1}^{m}[\tau_i : \sigma_i]},$$

but the un-collected version will be more convenient for our purposes.

Recall (from the statement of Theorem 8.4) that the vector space $\mathbf{C}_k^\Sigma(K)$ has as its basis the set of all k-dimensional Σ-critical simplices. We will define the desired linear maps from assertion (2) of Theorem 8.4 as matrices with respect to these chosen bases. And for each gradient path ρ as in (7), we indicate the first simplex σ_1 and last simplex τ_m by σ_ρ and τ_ρ respectively.

DEFINITION 8.7. For each dimension $k \geq 0$, the k-th **Morse boundary operator** is the linear map $d_k^\Sigma : \mathbf{C}_k^\Sigma(K) \to \mathbf{C}_{k-1}^\Sigma(K)$ given by the following matrix representation: its entry in the column of a critical k-simplex α and the row of a critical $(k-1)$-simplex ω is given by

$$[\alpha : \omega]_\Sigma = [\alpha : \omega] + \sum_\rho [\alpha : \sigma_\rho] \cdot w(\rho) \cdot [\tau_\rho : \omega], \tag{8}$$

where ρ ranges over all the Σ-paths.

There are three aspects of the formula (8) which might merit deeper consideration. First, the term $[\alpha : \omega]$ on the right side is precisely the entry in ω's column and α's row within the simplicial boundary matrix ∂_k^K — thus, the difference between this original entry and our new Σ-perturbed one is precisely the sum-over-paths term. Second, we don't have to sum over *all* the paths; the only paths that make a non-zero contribution are the ones which flow from α to ω like so:

$$\alpha \rhd \big(\sigma_1 \lhd \tau_1 \rhd \sigma_2 \lhd \tau_2 \rhd \cdots \rhd \sigma_m \lhd \tau_m\big) \rhd \omega.$$

And third, life gets much simpler when working over the field $\mathbb{F} = \mathbb{Z}/2$ because in this case each path connecting α to ω has weight 1; thus, it suffices to simply count the odd/even parity of the number of such connecting Σ-paths.

PROPOSITION 8.8. *The pair* $(\mathbf{C}_\bullet^\Sigma(K), d_\bullet^\Sigma)$ *constitutes a chain complex.*

PROOF. It suffices by induction to show that the desired result holds when Σ consists of a single pair $(\sigma \lhd \tau)$ of simplices in K; thus the set of critical simplices is $C_\Sigma = K - \{\sigma, \tau\}$, and the only Σ-path is

$$\rho = (\sigma \lhd \tau).$$

To show that $d_\bullet^\Sigma$ is a boundary operator, we must establish that for each fixed $\alpha, \omega \in C_\Sigma$, the sum

$$B = \sum_\xi [\alpha : \xi]_\Sigma \cdot [\xi : \omega]_\Sigma$$

equals zero when indexed over all $\xi \in C_\Sigma$. Using the formula (8), the contribution of each ξ to this sum is the product

$$B_\xi = \left([\alpha : \xi] - \frac{[\alpha : \sigma] \cdot [\tau : \xi]}{[\tau : \sigma]}\right) \cdot \left([\xi : \omega] - \frac{[\xi : \sigma] \cdot [\tau : \omega]}{[\tau : \sigma]}\right).$$

The negated term in the first factor disappears whenever $\dim \xi \neq \dim \sigma$, and the negated term in the second factor disappears whenever $\dim \xi \neq \dim \tau$. Thus, only three of the four terms survive when we multiply these two factors:

$$B_\xi = [\alpha : \xi] \cdot [\xi : \omega] - \frac{[\alpha : \sigma] \cdot [\tau : \xi] \cdot [\xi : \omega]}{[\tau : \sigma]} - \frac{[\alpha : \xi] \cdot [\xi : \sigma] \cdot [\tau : \omega]}{[\tau : \sigma]}$$

Summing over $\xi \in C_\Sigma$, we have $B = \sum_\xi B_\xi$ given by

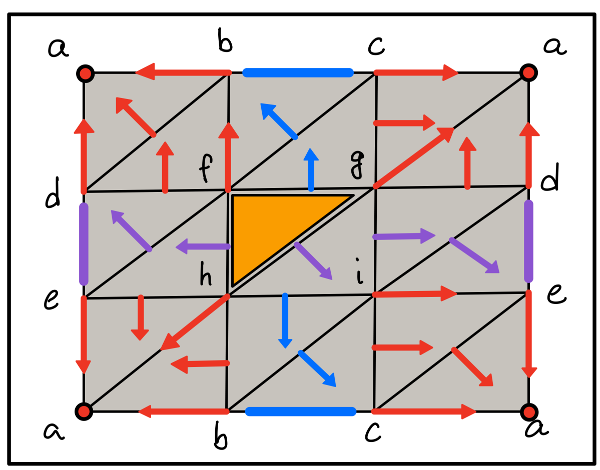
$$B = \sum_\xi [\alpha : \xi] \cdot [\xi : \omega] - \frac{[\alpha : \sigma]}{[\tau : \sigma]} \sum_\xi [\tau : \xi] \cdot [\xi : \omega] - \frac{[\tau : \omega]}{[\tau : \sigma]} \sum_\xi [\alpha : \xi] \cdot [\xi : \sigma].$$

It is now straightforward to check that $B = 0$ because $\partial^K_\bullet$ is a boundary operator on $\mathbf{C}_\bullet(K)$. In particular, the first sum evaluates to $-([\alpha : \sigma] \cdot [\sigma : \omega] + [\alpha : \tau] \cdot [\tau : \omega])$, while the second term evaluates to $[\alpha : \sigma] \cdot [\sigma : \omega]$ and the third term to $[\alpha : \tau] \cdot [\tau : \omega]$. □

As mentioned before, we call $(\mathbf{C}^\Sigma_\bullet(K), d^\Sigma_\bullet)$ the *Morse chain complex* associated to our acyclic partial matching Σ; although we have not yet shown that it has the same homology as $(\mathbf{C}_\bullet(K), \partial^K_\bullet)$, this is a good time to examine a few known cases and verify this assertion experimentally. One can build an acyclic partial matching on any simplicial complex by performing these two steps over and over until all simplices have been classified as matched or critical — initially, all simplices are unclassified:

(1) classify a simplex of lowest available dimension as critical; then,
(2) while there exist pairs $(\sigma \lhd \tau)$ of unclassified simplices so that σ is the only unclassified face of τ, classify $(\sigma \lhd \tau)$ as matched.

Although this process is not guaranteed to produce the largest acyclic partial matching (i.e., the one containing the fewest possible critical simplices), it is devastatingly effective in practice.



Illustrated here is the acyclic partial matching imposed by this simple two-step algorithm on the torus (note that the left and right edges of the figure have been identified, as have the top and bottom ones). In the first stage, one classifies the vertex a as critical; this creates various edges (such as ab, ad, etc.) with only one unclassified vertex in their boundaries — these produce the matchings indicated by red arrows. At the end of this process, all the vertices have been matched with edges, but there are several 2-simplices remaining with more than one unmatched edge in their boundaries. Next, we classify bc as critical and are allowed to make matchings indicated by the blue arrows. Next, we classify de as critical and make the purple matchings. Finally, only the simplex fgh remains unclassified, so it becomes critical. The critical simplices lie far away from each other, and do not form a subcomplex of the torus.

EXAMPLE 8.9. Let K be the triangulated torus and Σ the overlaid acyclic partial matching illustrated above. The Σ-critical simplices are $\{a, bc, de, fgh\}$, so the associated Morse chain complex has the form

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{F} \xrightarrow{d_2^\Sigma} \mathbb{F}^2 \xrightarrow{d_1^\Sigma} \mathbb{F} \longrightarrow 0$$

To really determine the boundary operators using (8) for arbitrary $\mathbb{F}$, we would have to impose an ordering on the vertices and keep careful track of minus signs. Let's instead work over $\mathbb{Z}/2$ and count gradient paths — there are two from bc to a, namely:

$$bc \rhd (b \lhd ab) \rhd a \quad \text{and} \quad bc \rhd (c \lhd ac) \rhd a.$$

Since there is an even number of connecting gradient paths, the entry $d_1^\Sigma|_{bc,a}$ equals 0. Proceeding similarly, one can check (exercise!) that both d_1^Σ and d_2^Σ are zero maps, which makes it trivial to compute the homology of the torus.

8.3 THE EQUIVALENCE

Let Σ be an acyclic partial matching on a simplicial complex K. Our goal here is to complete the proof of Theorem 8.4 by showing establishing the following result.

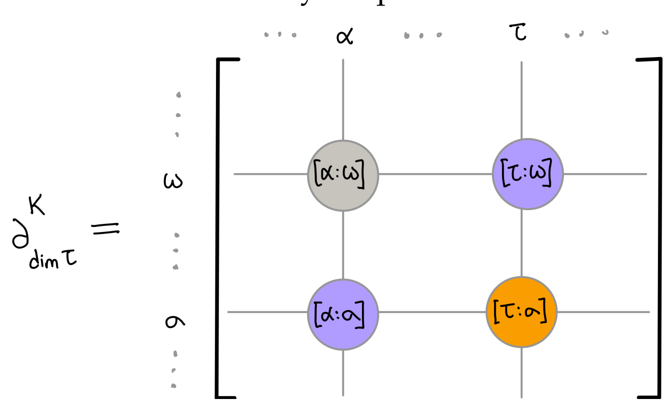
PROPOSITION 8.10. *The Morse chain complex $(\mathbf{C}_\bullet^\Sigma(K), d_\bullet^\Sigma)$ of Proposition 8.8 is chain homotopy equivalent to the standard simplicial chain complex $(\mathbf{C}_\bullet(K), \partial_\bullet^K)$.*

In other words, we will describe two chain maps

$$\psi_\bullet : \mathbf{C}_\bullet(K) \to \mathbf{C}_\bullet^\Sigma(K) \quad \text{and} \quad \phi_\bullet : \mathbf{C}_\bullet^\Sigma(K) \to \mathbf{C}_\bullet(K)$$

along with a pair of chain homotopies relating $\phi_\bullet \circ \psi_\bullet$ and $\psi_\bullet \circ \phi_\bullet$ to the identity chain maps on $\mathbf{C}_\bullet(K)$ and $\mathbf{C}_\bullet^\Sigma(K)$ respectively. The best way to build $\psi_\bullet$ and $\phi_\bullet$ is by processing the simplex-pairs $(\sigma \lhd \tau)$ in Σ one at a time. Given this strategy, it is instructive to first examine the special case where Σ contains a single pair $(\sigma \lhd \tau)$.

Consider the entries (in the usual matrix representation) of $\partial_{\dim \tau}^K$ corresponding not only to our chosen pair $(\sigma \lhd \tau)$, but also two arbitrary simplices α and ω.



In order to algebraically disentangle σ and τ from the other simplices, we treat the ± 1 entry $[\tau : \sigma]$ as a pivot and seek to clear out all the other entries in both $\mathrm{Col}(\tau)$ and $\mathrm{Row}(\sigma)$. This requires performing row and column operations of the form

$$\mathrm{Row}(\omega) \leftarrow \mathrm{Row}(\omega) - \frac{[\tau : \omega]}{[\tau : \sigma]} \cdot \mathrm{Row}(\sigma) \quad \Big| \quad \mathrm{Col}(\alpha) \leftarrow \mathrm{Col}(\alpha) - \frac{[\alpha : \sigma]}{[\tau : \sigma]} \cdot \mathrm{Col}(\tau). \tag{9}$$

After these operations have been performed, the entry in α's column and ω's row equals

$$[\alpha : \omega] + [\alpha : \sigma] \cdot \frac{-1}{[\tau : \sigma]} \cdot [\tau : \omega], \tag{10}$$

which agrees with the expression for $[\alpha : \omega]_\Sigma$ from (8) because there is only one Σ-path $\sigma \lhd \tau$. More importantly, the row and column operations of (9) suggest the structure of the desired chain maps which take us from $\mathbf{C}_\bullet(K)$ to $\mathbf{C}^\Sigma_\bullet(K)$ and back. This allows us to prove Proposition 8.10 in the special case where Σ contains only one pair.

LEMMA 8.11. *Let Σ be an acyclic partial matching on K containing only one pair $(\sigma \lhd \tau)$. Then the simplicial chain complex $(\mathbf{C}_\bullet(K), \partial^K_\bullet)$ is chain homotopy equivalent to the Morse complex $(\mathbf{C}^\Sigma_\bullet(K), d^\Sigma_\bullet)$.*

PROOF. For each $k \geq 0$, define the linear maps $\psi_k : \mathbf{C}_k(K) \to \mathbf{C}^\Sigma_k(K)$ by the following matrix representation; for each pair of k-simplices (α, ω) in $K \times (K - \{\sigma, \tau\})$, the entry in α's column and ω's row is

$$\psi_k\big|_{\alpha,\omega} = \begin{cases} -\frac{[\tau:\omega]}{[\tau:\sigma]} & \alpha = \sigma \\ 1 & \alpha = \omega \neq \tau \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Conversely, define the linear maps $\phi_k : \mathbf{C}^\Sigma_k(K) \to \mathbf{C}_k(K)$ by placing the following entry in the column of ω in $K - \{\sigma, \tau\}$ and the row of α in K:

$$\phi_k\big|_{\omega,\alpha} = \begin{cases} -\frac{[\omega:\sigma]}{[\tau:\sigma]} & \alpha = \tau \\ 1 & \omega = \alpha \neq \sigma \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Checking that both $\psi_\bullet$ and $\phi_\bullet$ are chain maps has been relegated to two of the Exercises. To extract the chain homotopies, first note that $\psi_\bullet \circ \phi_\bullet$ equals the identity map on $\mathbf{C}^\Sigma_\bullet(K)$. Conversely, the composite $\phi_\bullet \circ \psi_\bullet$ is given by

$$\phi_k \circ \psi_k\big|_{\alpha,\alpha'} = \begin{cases} -\frac{[\tau:\alpha']}{[\tau:\sigma]} & \alpha = \tau \neq \alpha' \\ -\frac{[\alpha:\sigma]}{[\tau:\sigma]} & \alpha \neq \sigma = \alpha' \\ 1 & \alpha = \alpha' \\ 0 & \text{otherwise.} \end{cases}$$

One can now check that the linear maps $\theta_k : \mathbf{C}_k(K) \to \mathbf{C}_{k+1}(K)$ prescribed by

$$\theta\big|_{\alpha,\beta} = \begin{cases} \frac{1}{[\tau:\sigma]} & \alpha = \sigma \text{ and } \beta = \tau \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

furnish the desired chain homotopy between $\phi_k \circ \psi_k$ and the identity chain map. ☐

The acyclicity of Σ plays an important role when attempting to iteratively apply Lemma 8.11 for the purposes of proving Proposition 8.10. Acyclicity guarantees that removing a single pair $(\sigma \lhd \tau) \in \Sigma$ from K does not alter the entry $[\tau' : \sigma']$ in the boundary matrix corresponding to another pair $(\sigma' \lhd \tau') \in \Sigma$. To see why, note from (10) that the difference between the old and new entries equals

$$\frac{[\tau' : \sigma] \cdot [\tau : \sigma']}{[\tau : \sigma]}.$$

Assuming that the numerator is nonzero, we are forced to conclude that the the Σ-path $\sigma \lhd \tau \rhd \sigma' \lhd \tau'$ is not gradient, which leads to the desired contradiction. As a consequence, the repeated application of Lemma 8.11 correctly converges to the Morse complex regardless of the order in which we remove the simplex-pairs lying in Σ.

8.4 For Persistence

The machinery of acyclic partial matchigns and Morse complexes is extremely flexible, and admits powerful generalizations. Here we will describe how to construct filtered Morse complexes for the purposes of simplifying the persistent homology computations which formed the focus of Chapter 6. Let $\boldsymbol{F}_\bullet K$ be a ($\mathbb{R}_+$-indexed) filtration of a simplicial complex K, and let $b : K \to \mathbb{R}_+$ be the associated monotone function $\sigma \mapsto \inf\{t \geq 0 \mid \sigma \in \boldsymbol{F}_t(K)\}$.

DEFINITION 8.12. An acyclic partial matching Σ on K is **$\boldsymbol{F}_\bullet$-compatible** if $b(\sigma) = b(\tau)$ holds for every pair of simplices $(\sigma \lhd \tau)$ in Σ.

This compatibility requirement forces Σ-paths to be decreasing with respect to b.

PROPOSITION 8.13. *Let Σ be an $\boldsymbol{F}_\bullet$-compatible acyclic partial matching on K. For any Σ-path*

$$\rho = \sigma_1 \lhd \tau_1 \rhd \cdots \rhd \sigma_m \lhd \tau_m,$$

we have $b(\sigma_i) \geq b(\sigma_j)$ for all $i \leq j$.

PROOF. For each $i \in \{1, \ldots, m\}$ we have an equality $b(\sigma_i) = b(\tau_i)$ by the $\boldsymbol{F}_\bullet$-compatibility of Σ and an inequality $b(\tau_i) \geq b(\sigma_{i+1})$ by the monotonicity of $b : K \to \mathbb{R}$. $\square$

This elementary observation has some wonderful consequences when it comes to simplifying computations of persistent homology. For each $t \in \mathbb{R}_+$, let $\Sigma_t \subset \Sigma$ be the restriction of Σ to (pairs which lie in) the subcomplex $\boldsymbol{F}_t K \subset K$, and let $(M^t_\bullet, d^t_\bullet)$ be shorthand for the affiliated Morse complex $(\mathbf{C}^{\Sigma_t}_\bullet(\boldsymbol{F}_t K), \partial^{\Sigma_t}_\bullet)$.

COROLLARY 8.14. *For each pair $0 \leq t \leq s$ of real numbers, there is an inclusion $(M^t_\bullet, d^t_\bullet) \hookrightarrow (M^s_\bullet, d^s_\bullet)$ of Morse chain complexes.*

PROOF. The critical simplices in $\boldsymbol{F}_t K$ remain critical in $\boldsymbol{F}_s K$, so M^t_k is naturally a subspace of M^s_k for all $k \geq 0$. Thus, it suffices to check that the Morse boundary operator d^s_k equals d^t_k when restricted to the subspace M^t_k. But this follows directly from the formula (8) — consider a Σ-critical k-simplex $\alpha \in \boldsymbol{F}_t K$, and a Σ-path of the form

$$\rho = (\sigma_1 \lhd \tau_1 \rhd \cdots \rhd \sigma_m \lhd \tau_m)$$

so that $\alpha \rhd \sigma_1$. By the monotonicity of b, we have $t \geq b(\alpha) \geq \sigma_1$. Now Proposition 8.13 guarantees that all subsequent Σ-paired simplices $\sigma_i \lhd \tau_i$ appearing in ρ must have b-values bounded above by t. In particular, adding new simplices from $(\boldsymbol{F}_s - \boldsymbol{F}_t)$ can not possibly change the Σ-paths over which we sum when evaluating the Morse boundary of α in M^s_k, whence $d^s_k(\alpha) = d^t_k(\alpha)$ as desired. $\square$

Having found a nested sequence of Morse complexes, one seeks to relate persistent homology groups of $\mathbf{H}_k(\boldsymbol{F}_\bullet K)$ to those of $\mathbf{H}_k(M^\bullet, d^\bullet)$. The basic idea, as one might expect, is to unite all the chain homotopy equivalences $\{\psi_t, \phi_t \mid t \geq 0\}$ promised by Proposition 8.10 between $\mathbf{C}_\bullet(\boldsymbol{F}_t K)$ and M^t for each $t \geq 0$ into a single equivalence relating the two persistence modules.

THEOREM 8.15. *For each dimension $k \geq 0$ and pair of real numbers $0 \leq t \leq s$, there are isomorphisms*
$$\mathbf{PH}_{t\to s}\mathbf{H}_k(\boldsymbol{F}_\bullet K) \simeq \mathbf{PH}_{t\to s}\mathbf{H}_k(M^\bullet, d^\bullet)$$
of persistent homology groups. Therefore, the barcodes of $\mathbf{H}_k(\boldsymbol{F}_\bullet K)$ and $\mathbf{H}_k(M^\bullet, d^\bullet)$ are equal.

PROOF. Enumerate all the simplex-pairs in Σ according to their b-values, i.e., write
$$\Sigma = \{(\sigma_1 \lhd \tau_1), (\sigma_2 \lhd \tau_2), \ldots, (\sigma_m \lhd \tau_m)\}$$
so that $b(\sigma_i) \leq b(\sigma_j)$ whenever $i \leq j$. Applying Lemma 8.11 to the Σ-pairs in this order, we obtain a family of chain homotopy equivalences indexed by $t \geq 0$:
$$\psi^t_\bullet : \mathbf{C}_\bullet(\boldsymbol{F}_t K) \to M^t_\bullet \quad \text{and} \quad \phi^t_\bullet : M^t_\bullet \to \mathbf{C}_\bullet(\boldsymbol{F}_t K)$$
which fit into a commuting diagram with the natural inclusion maps. Namely, for any pair of positive real numbers $t \leq s$ and dimension $k \geq 0$, the following diagrams of vector spaces commute:
$$\begin{tikzcd} \mathbf{C}_k(\boldsymbol{F}_t K) \arrow[r, hook] \arrow[d, "\psi^t_k"'] & \mathbf{C}_k(\boldsymbol{F}_s K) \arrow[d, "\psi^s_k"] \\ M^t_k \arrow[r, hook] & M^s_k \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C}_k(\boldsymbol{F}_t K) \arrow[r, hook] & \mathbf{C}_k(\boldsymbol{F}_s K) \\ M^t_k \arrow[r, hook] \arrow[u, "\phi^t_k"] & M^s_k \arrow[u, "\phi^s_k"'] \end{tikzcd}$$
Since ψ^t and ϕ^t form two halves of a chain homotopy equivalence, they induce isomorphisms on k-th homology for all $k \geq 0$. Thus, we obtain a 0-interleaving between the two k-th homology persistence modules, which guarantees that all their persistent homology groups are isomorphic. □

From the perspective of using this result to simplify computations, it is important to note that large $\boldsymbol{F}_\bullet$-compatible partial matchings can only be found on filtrations where lots of simplices share the same b-values. Fortunately, this requirement is always satisfied by the Vietoris-Rips filtration. Consider a collection of points $P = \{p_0, \ldots, p_k\}$ so that the largest pairwise distance $d(p_i, p_j)$ equals $t' > 0$, corresponding to a single edge (p_i, p_j). Then the set of simplices born at this scale t' in $\mathbf{VR}_\bullet(P)$ include not only our edge, but also every other simplex containing this edge in its boundary.

8.5 FOR SHEAVES

Aside from the usual cognitive dissonance caused by reversing arrows when transitioning from homology to cohomology, there are not too many obstacles involved in using acyclic partial matchings to simplify sheaf cohomology computations. Let $\mathscr{S}$ be a sheaf (see Definition 7.1) on a simplicial complex K.

DEFINITION 8.16. An acyclic partial matching Σ on K is $\mathscr{S}$**-compatible** if the restriction map $\mathscr{S}(\sigma \leq \tau)$ is an isomorphism for every pair $(\sigma \lhd \tau)$ in Σ.

The weights of gradient paths from Definition 8.6 must now be upgraded from scalars to linear maps. It will be convenient, for simplices α, β in K, to define the *scaled restriction map*

$\mathscr{S}_{\alpha,\beta} : \mathscr{S}(\alpha) \to \mathscr{S}(\beta)$ as

$$\mathscr{S}_{\alpha,\beta} = [\beta : \alpha] \cdot \mathscr{S}(\alpha \leq \beta) = \begin{cases} +\mathscr{S}(\alpha \leq \beta) & \alpha = \beta_{-i} \text{ for even } i, \\ -\mathscr{S}(\alpha \leq \beta) & \alpha = \beta_{-i} \text{ for odd } i, \\ 0 & \text{otherwise.} \end{cases}$$

This linear map forms the block in α's column and β's row in the coboundary operator $\partial^{\mathscr{S}}_{\bullet}$ from Definition 7.6. For each Σ-path

$$\rho = (\sigma_1 \lhd \tau_1 \rhd \sigma_2 \lhd \tau_2 \rhd \cdots \rhd \sigma_m \lhd \tau_m);$$

define the $\mathscr{S}$**-weight** $w_{\mathscr{S}}(\rho)$ to be the composite linear map $\mathscr{S}(\tau_m) \to \mathscr{S}(\sigma_1)$ given by

$$(-1)^m \cdot \left[\mathscr{S}^{-1}_{\sigma_1,\tau_1} \circ \mathscr{S}_{\sigma_2,\tau_1} \circ \mathscr{S}^{-1}_{\sigma_2,\tau_2} \circ \cdots \circ \mathscr{S}^{-1}_{\sigma_m,\tau_m}\right].$$

Unsurprisingly, these $\mathscr{S}$-weights make an appearance when defining the Morse complex of Σ with $\mathscr{S}$-coefficients.

DEFINITION 8.17. Let $\mathscr{S}$ be a sheaf over the simplicial complex K and Σ an $\mathscr{S}$-compatible acyclic partial matching. The **Morse complex of Σ with coefficients in $\mathscr{S}$** is a cochain complex

$$(\mathbf{C}^{\bullet}_{\Sigma}(K;\mathscr{S}), \partial^{\bullet}_{\mathscr{S},\Sigma})$$

defined as follows. For each dimension $k \geq 0$,

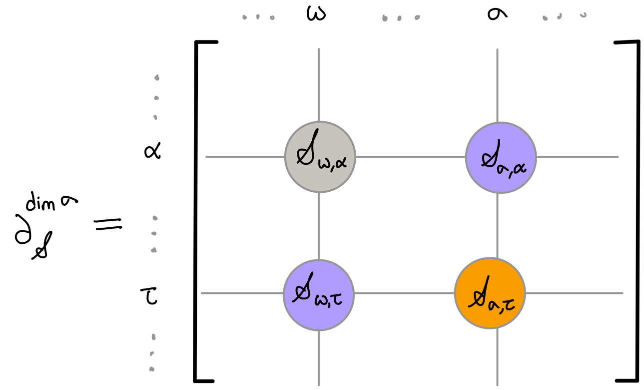
(1) the vector space $\mathbf{C}^k_{\Sigma}(K;\mathscr{S})$ equals the product of stalks $\prod_{\alpha} \mathscr{S}(\alpha)$ where α ranges over the k-dimensional Σ-critical simplices, and
(2) the linear map $\partial^k_{\mathscr{S},\Sigma} : \mathbf{C}^k_{\Sigma}(K;\mathscr{S}) \to \mathbf{C}^{k+1}_{\Sigma}(K;\mathscr{S})$ is represented by a block-matrix whose entry in α's column and ω's row equals

$$\partial^k_{\mathscr{S},\Sigma}\Big|_{\alpha,\omega} = \mathscr{S}_{\alpha,\omega} + \sum_{\rho} \mathscr{S}_{\sigma_\rho,\omega} \circ w_{\mathscr{S}}(\rho) \circ \mathscr{S}_{\alpha,\tau_\rho},$$

where ρ ranges over all the Σ-paths.

The fact that this definition actually produces a cochain complex follows from arguments analogous to the ones which we used in the proof of Proposition 8.8; the most significant difference is that unlike scalars of the form $[\alpha : \omega]$ used throughout that proof, the linear maps $\mathscr{S}_{\alpha,\omega}$ do not (necessarily) commute with each other.

Similarly, all the results of Section 3 admit direct generalizations to the sheafy context, with two caveats. First, we are working with cohomology rather than homology, so the boundary matrix is transposed. And second, we are working with an arbitrary sheaf, so the coboundary matrix is populated by block sub-matrices rather than scalar entries. For each $(\sigma \lhd \tau)$ in Σ, the motivating picture is provided by the usual matrix representation of the coboundary $\partial^{\dim\sigma}_{\mathscr{S}}$:



From this picture, one can discover the row and column operations that are required to turn the (invertible!) block $\mathscr{S}_{\sigma,\tau}$ into a pivot, and hence deduce the cochain homotopy equivalences which form counterparts of the maps ψ and ϕ from Lemma 8.11. Here is the aftermath.

THEOREM 8.18. *Let $\mathscr{S}$ be a sheaf on a simplicial complex K and let Σ be a $\mathscr{S}$-compatible acyclic partial matching on K. Then for each dimension $k \geq 0$, the sheaf cohomology group $\mathbf{H}^k(K;\mathscr{S})$ is isomorphic to the k-th cohomology group of the Morse cochain complex $(\mathbf{C}^\bullet_\Sigma(K;\mathscr{S}), \partial^\bullet_{\mathscr{S},\Sigma})$.*

The advantage of using the Morse complex in practice for computing sheaf cohomology is that it tends to be much smaller, since the cochain groups are built using stalks of the critical simplices (rather than all simplices). On the other hand, the compatibility requirement on Σ is quite severe — to find large acyclic partial matchings which happen to be compatible with a sheaf, we require the presence of many simplex-pairs $(\sigma \lhd \tau)$ for which the associated restriction map is invertible.

EXERCISES

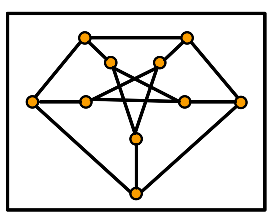
EXERCISE 8.1. Let Σ be an acyclic partial matching on a simplicial complex K. Show that the Euler characteristic of K is given by

$$\chi(K) = \sum_{k=0}^{\dim K} (-1)^k \cdot m_k,$$

where m_k is the number of k-dimensional Σ-critical simplices.

EXERCISE 8.2. Write down all the gradient paths between critical simplices in Example 8.9 and confirm that the Morse chain complex has zero boundary operators over $\mathbb{Z}/2$.

EXERCISE 8.3. When not functioning as an occult symbol, the **Petersen graph** serves as the source of many counterexamples in graph theory.



Impose an acyclic partial matching on this graph and use it to compute the homology groups over $\mathbb{Z}/2$ without performing any matrix operations.

EXERCISE 8.4. Show that the maps $\psi_\bullet : \mathbf{C}_\bullet(K) \to \mathbf{C}^\Sigma_\bullet(K)$ defined in (11) form a chain map .

EXERCISE 8.5. Show that the maps $\phi_\bullet : \mathbf{C}^\Sigma_\bullet(K) \to \mathbf{C}_\bullet(K)$ defined in (12) form a chain map.

EXERCISE 8.6. Show that the maps $\theta_k : \mathbf{C}_k(K) \to \mathbf{C}_{k+1}(K)$ from (13) serve as a chain homotopy between $\phi_\bullet \circ \psi_\bullet$ and the identity chain map on $\mathbf{C}_k(K)$.

EXERCISE 8.7. Verify that the two diagrams in the proof of Theorem 8.15 actually commute.

EXERCISE 8.8. State and prove a version of Lemma 8.11 in the context of a sheaf $\mathscr{S}$ on a simplicial complex K equipped with an $\mathscr{S}$-compatible acyclic partial matching Σ.